Correctness of Linear Search

Theorem: If, on input, L is an array of $n \geq 0$ items indexed from 1 to $n$ and $x$ is the item sought, then algorithm l_search terminates with index equal to the index of the first occurrence of $x$ in $L$, if $x$ is in $L$, and equal to 0 otherwise.

Proof: First we show by induction on $k$ that, for $1 \leq k \leq n+1$, if and when control reaches the tests in line 2 for the $k$-th time, the following conditions (loop invariants) are satisfied: index = $k$ and for $1 \leq i < k$, $L[i] \neq x$.

Base Case. If $k = 1$, then index = $k$ from line 1, and the second condition is vacuously satisfied.

Induction Hypothesis: Now assume that the conditions are satisfied for some $k < n+1$.

Induction Step: Show the conditions are satisfied for $k+1$. By the induction hypothesis, $L[i] \neq x$ for $1 \leq i \leq k$, and index = $k$ when line 2 was executed the $k$-th time. If the tests on line 2 are executed again, that is, the $(k+1)$-st time, we conclude that the tests were satisfied the $k$-th time, and the body of the loop was executed, so $L[k] \neq x$. Also, index is incremented in the loop, so the $(k+1)$-st time the tests at line 2 are executed, the required conditions hold. This ends the induction proof.

Now, suppose that the tests in line 2 are executed exactly $k$ times. Clearly, $1 \leq k \leq n+1$. Consider the two possible cases when line 5 is executed after the loop. The output is 0 if and only if $k = n+1$. By the statement we just proved, for $1 \leq i < n+1$, $L[i] \neq x$; that is, $x$ is not in array $L$, so the output is correct. (Note that this includes the case where $n=0$ and the list is empty.)

On the other hand, the output is index = $k$, where $1 < k \leq n$, if and only if the loop terminated because $L[k] = x$. Since for $1 \leq i < k$, $L[i] \neq x$, we conclude that $k$ is the index of the first occurrence of $x$ in the array.