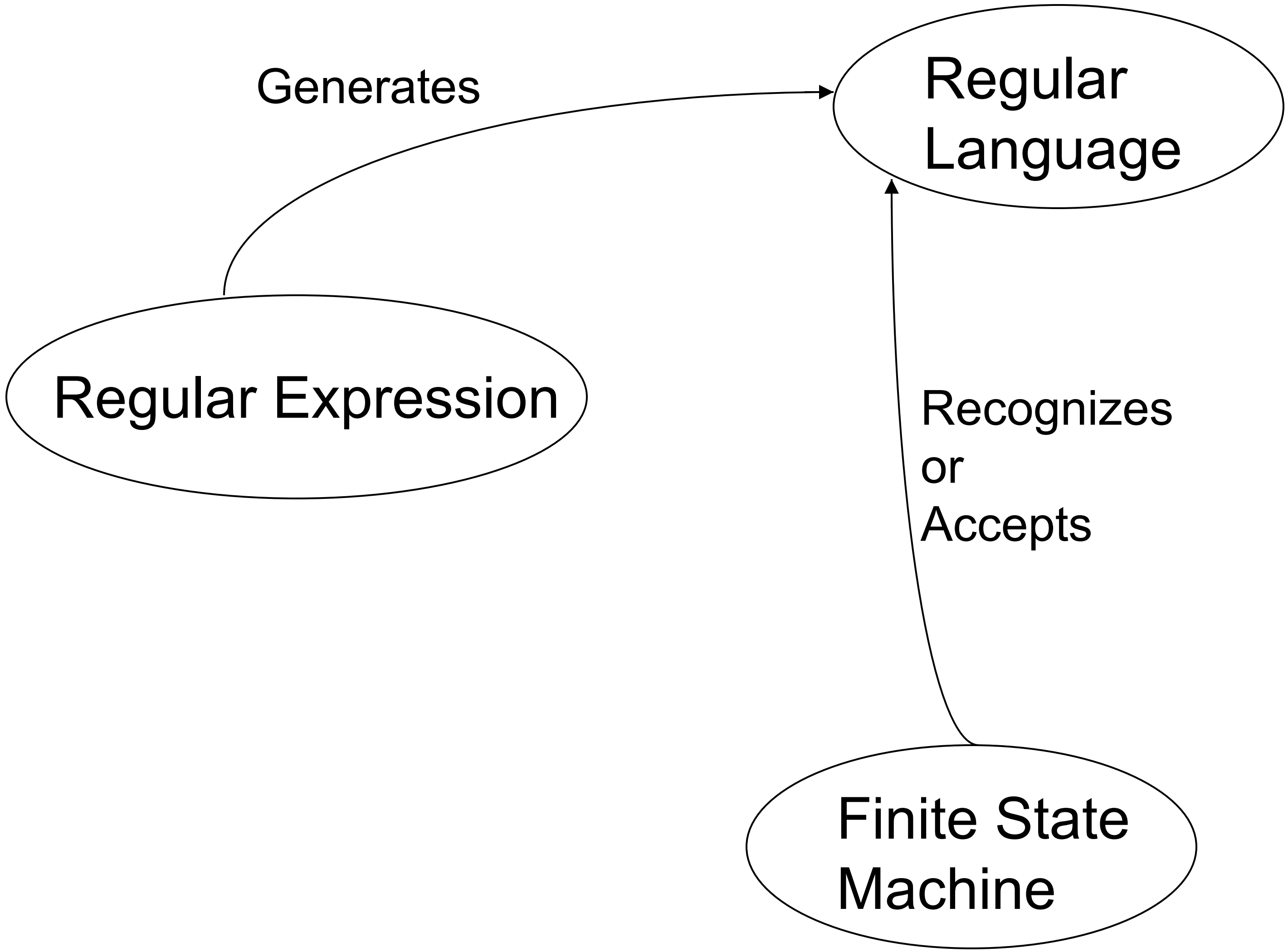
Regular Expressions

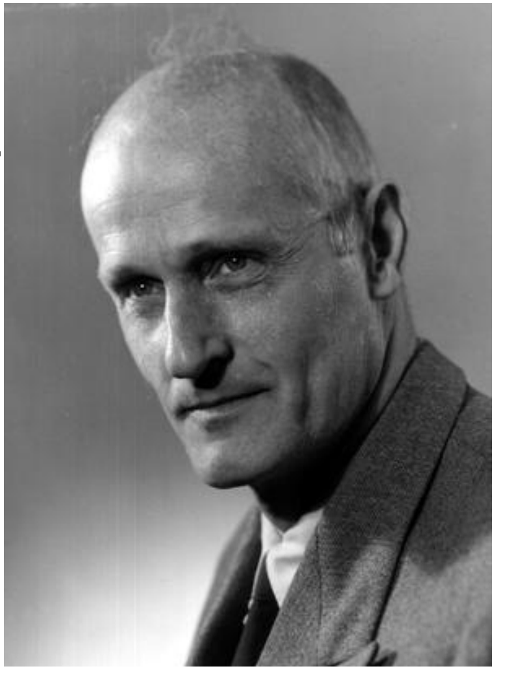
Chapter 6

Regular Languages



Stephen Cole Kleene

- 1909 – 1994, mathematical logician
- One of many distinguished students (e.g., Alan Turing) of Alonzo Church (lambda calculus) at Princeton.
- Best known as a founder of the branch of mathematical logic known as recursion theory.
- Also invented regular expressions.
- Kleene pronounced his last name *KLAY-nee*. *`kli:ni* and *`kli:n* are common mispronunciations.
  - His son, Ken Kleene, wrote: "As far as I am aware this pronunciation is incorrect in all known languages. I believe that this novel pronunciation was invented by my father. "



- Kleeneness is next to Godelness
  - Cleanliness is next to Godliness

Regular Expressions

Regular expression Σ contains two kinds of symbols:

- special symbols, $\varnothing$, ε, *, +, $\cup$, (,) …
- symbols that regular expressions will match against

The regular expressions over an alphabet Σ are all and only the strings that can be obtained as follows:

1. $\varnothing$ is a regular expression.
2. ε is a regular expression.
3. Every element of Σ is a regular expression.
4. If α , β are regular expressions, then so is $\alpha\beta$.
5. If α , β are regular expressions, then so is $\alpha\cup\beta$.
6. If α is a regular expression, then so is α^*.
7. α is a regular expression, then so is α^+.
8. If α is a regular expression, then so is (α).

Regular Expression Examples

If $\Sigma = \{a, b\}$, the following are regular expressions:

$\varnothing$

ε

a

$(a \cup b)^*$

$abba \cup \varepsilon$

Regular Expressions Define Languages

- Regular expressions are useful because each RE has a meaning
- If the meaning of an RE α is the language *A*, then we say that α defines or describes *A*.

Define *L*, a **semantic interpretation function** for regular expressions:

1. $L(\varnothing) = \varnothing$. //the language that contains no strings
2. $L(\varepsilon) = \{\varepsilon\}$. //the language that contains just the empty string
3. $L(c) = \{c\}$, where $c \in \Sigma$.
4. $L(\alpha\beta) = L(\alpha)\, L(\beta)$.
5. $L(\alpha \cup \beta) = L(\alpha) \cup L(\beta)$.
6. $L(\alpha^*) = (L(\alpha))^*$.
7. $L(\alpha^+) = L(\alpha\alpha^*) = L(\alpha)\,(L(\alpha))^*$. If $L(\alpha)$ is equal to $\varnothing$, then $L(\alpha^+)$ is also equal to $\varnothing$. Otherwise $L(\alpha^+)$ is the language that is formed by concatenating together one or more strings drawn from $L(\alpha)$.
8. $L((\alpha)) = L(\alpha)$.

The Role of the Rules

- Rules 1, 3, 4, 5, and 6 give the language its power to define sets.
- Rule 8 has as its only role grouping other operators.
- Rules 2 and 7 appear to add functionality to the regular expression language, but they don't.

2. ε is a regular expression.

7. α is a regular expression, then so is α^+.

Analyzing a Regular Expression

The compositional semantic interpretation function lets us map between regular expressions and the languages they define.

$$L((\mathtt{a} \cup \mathtt{b})^*\mathtt{b}) = L((\mathtt{a} \cup \mathtt{b})^*)\ L(\mathtt{b})$$

$$= (L((\mathtt{a} \cup \mathtt{b})))^*\ L(\mathtt{b})$$

$$= (L(\mathtt{a}) \cup L(\mathtt{b}))^*\ L(\mathtt{b})$$

$$= (\{\mathtt{a}\} \cup \{\mathtt{b}\})^*\ \{\mathtt{b}\}$$

$$= \{\mathtt{a}, \mathtt{b}\}^*\ \{\mathtt{b}\}.$$

Examples

$L($ a*b* $) =$

$L($ (a ∪ b)* $) =$

$L($ (a ∪ b)*a*b* $) =$

$L($ (a ∪ b)*abba(a ∪ b)* $) =$

$L($ (a ∪ b) (a ∪ b)a(a ∪ b)* $) =$

Going the Other Way

Given a language, find a regular expression

$L = \{w \in \{a, b\}^*: |w| \text{ is even}\}$

$((a \cup b)\ (a \cup b))^*$

$(aa \cup ab \cup ba \cup bb)^*$

$L = \{w \in \{a, b\}^*: w \text{ contains an odd number of a's}\}$

$b^*\ (ab^*ab^*)^*\ a\ b^*$

$b^*\ a\ b^*\ (ab^*ab^*)^*$

Common Idioms

$(\alpha \cup \varepsilon)$

- Optional α, matching α or the empty string

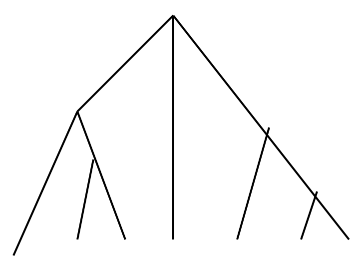
$(\mathtt{a} \cup \mathtt{b})^*$

- Set of all strings composed of the characters a and b

- The regular expression a* is simply a string. It is different from the language L(a*) ={w: w is composed of zero or more a's}.
- However, when no confusion, we do not write the semantic interpretation function explicitly. We will say things like, "The language a* is infinite"

Operator Precedence in Regular Expressions

	Regular Expressions	Arithmetic Expressions
Highest	Kleene star	exponentiation
↕	concatenation	multiplication
Lowest	union	addition



$a\ b^* \cup c\ d^*$ $\qquad$ $x\ y^2 + i\ j^2$

Details Matter

$a^* \cup b^* \neq (a \cup b)^*$

$(ab)^* \neq a^*b^*$

Kleene's Theorem

Finite state machines and regular expressions define the same class of languages. To prove this, we must show:

Theorem: Any language that can be defined with a regular expression can be accepted by some FSM and so is regular.

Theorem: Every regular language (i.e., every language that can be accepted by some DFSM) can be defined with a regular expression.

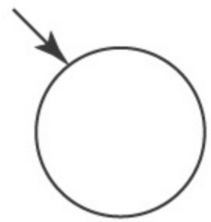
- Sometimes FSM is easy, sometimes RE is easy.

For Every Regular Expression α, There is a Corresponding FSM M s.t. $L(\alpha) = L(M)$

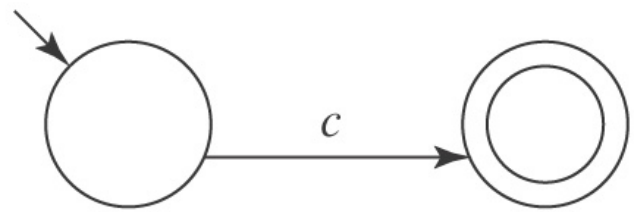
We'll show this by construction.

First, primitive regular expressions, then regular expressions that exploit the operations of union, concatenation, and Kleene star.

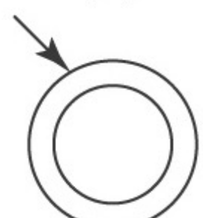
$\varnothing$:



A single element of Σ:

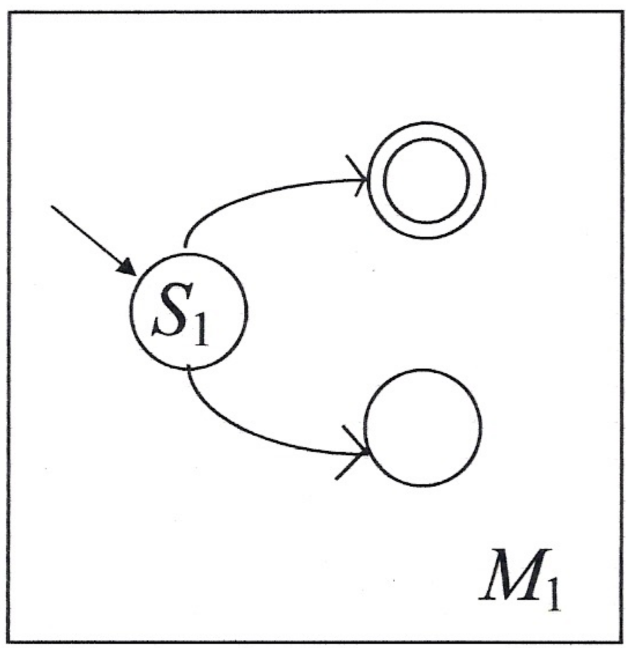


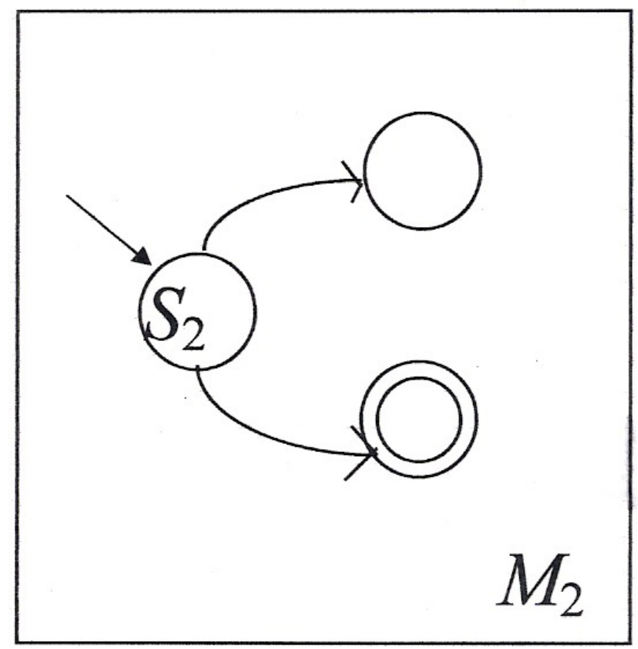
ε ($\varnothing^*$):



Union

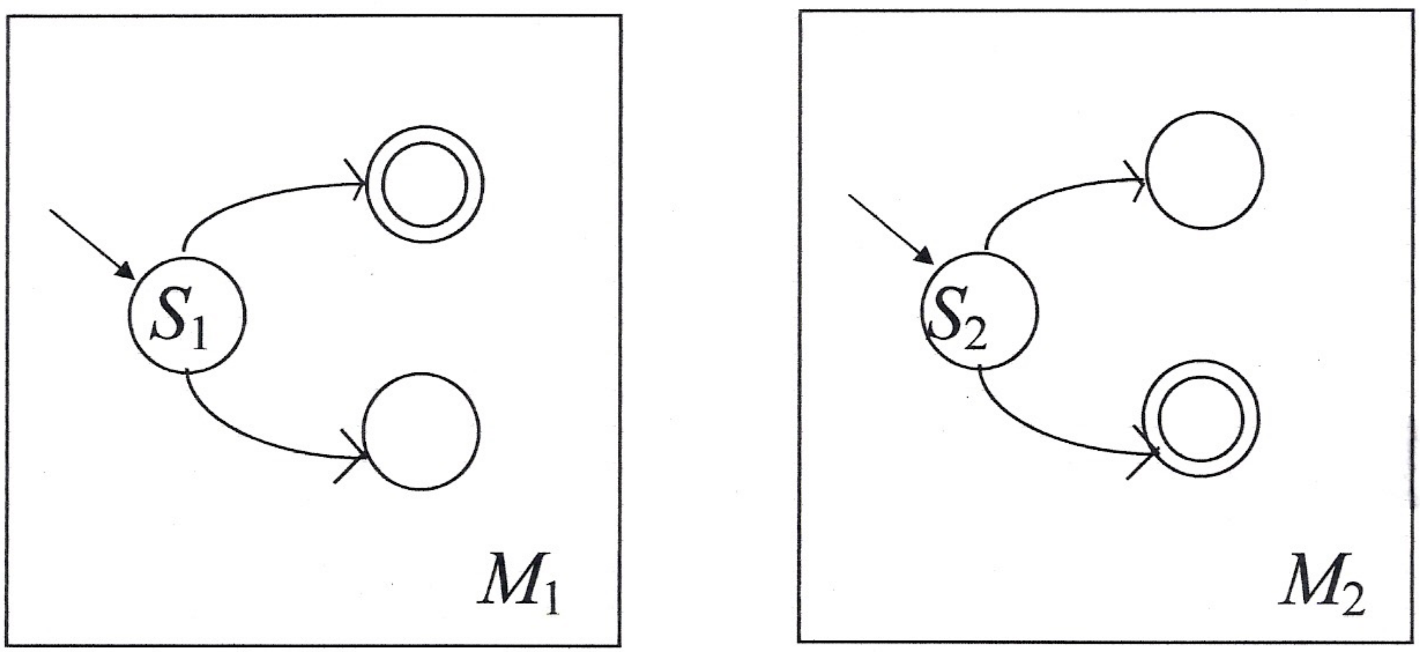
If α is the regular expression $\beta \cup \gamma$ and if both $L(\beta)$ and $L(\gamma)$ are regular:





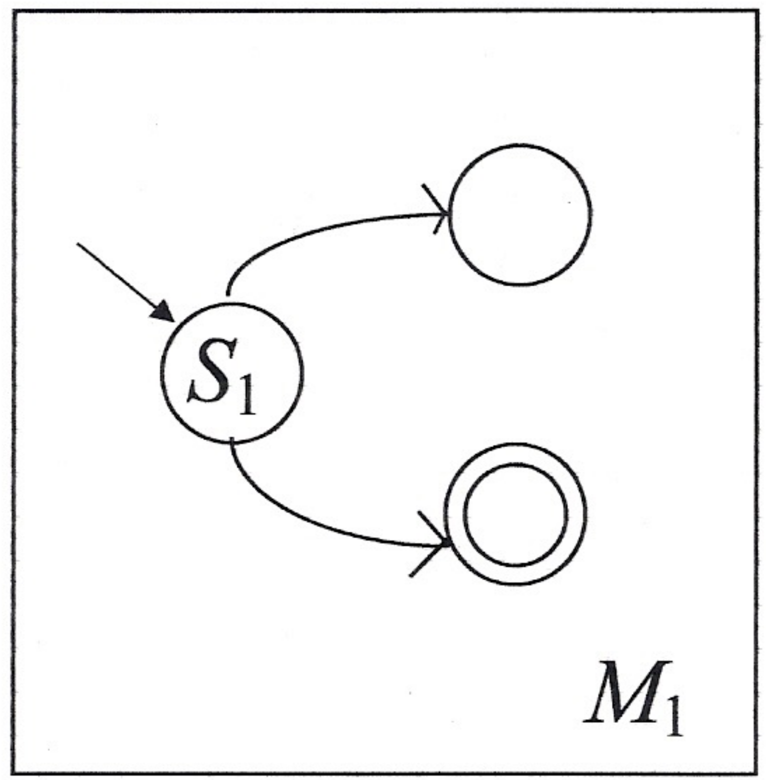
Concatenation

If α is the regular expression $\beta\gamma$ and if both $L(\beta)$ and $L(\gamma)$ are regular:



Kleene Star

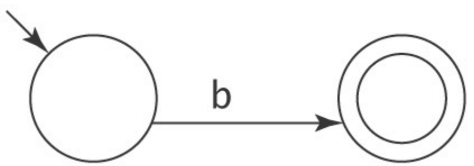
If α is the regular expression β^* and if $L(\beta)$ is regular:



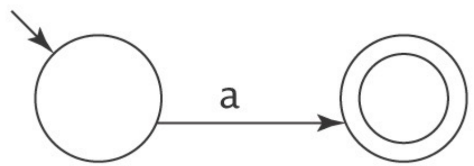
From RE to FSM: An Example

(b ∪ ab)*

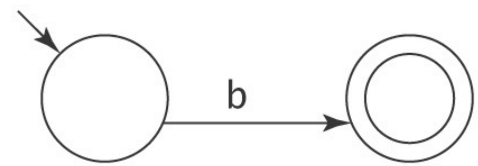
An FSM for b



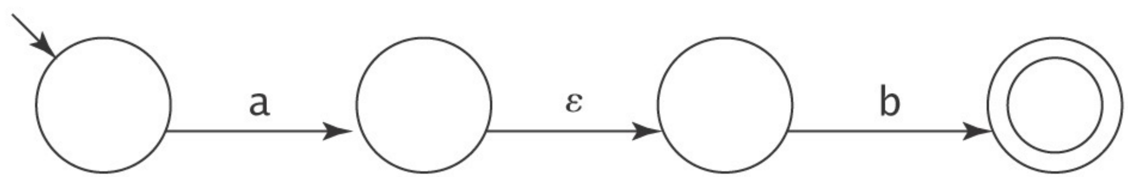
An FSM for a



An FSM for b



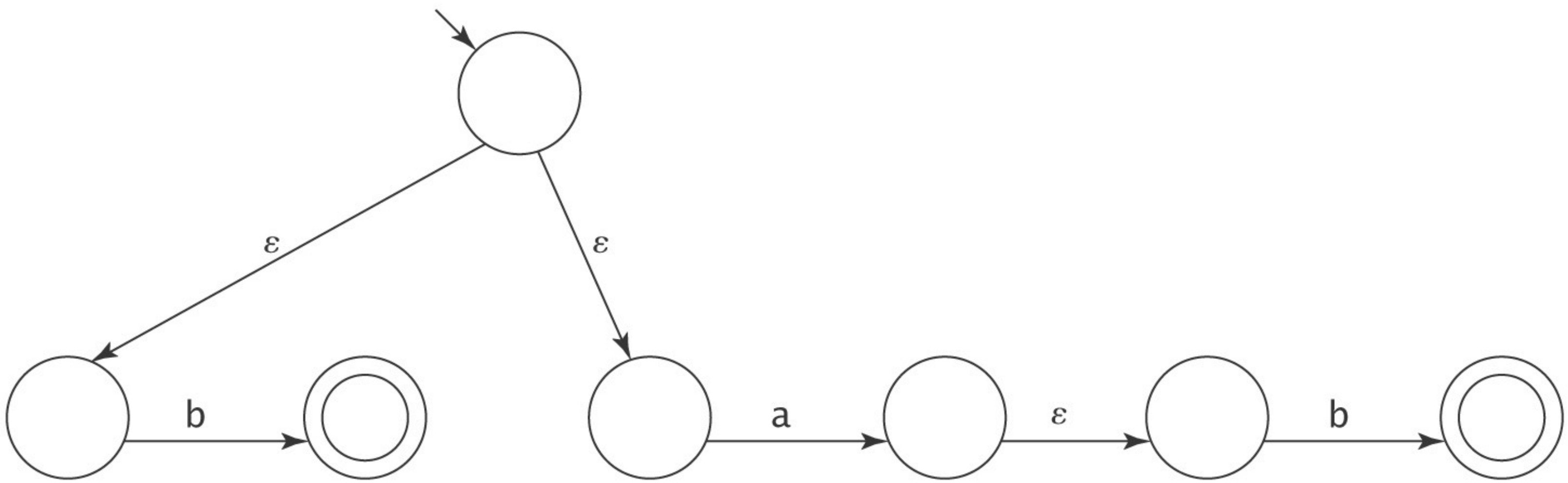
An FSM for ab:



An Example

(b ∪ ab)*

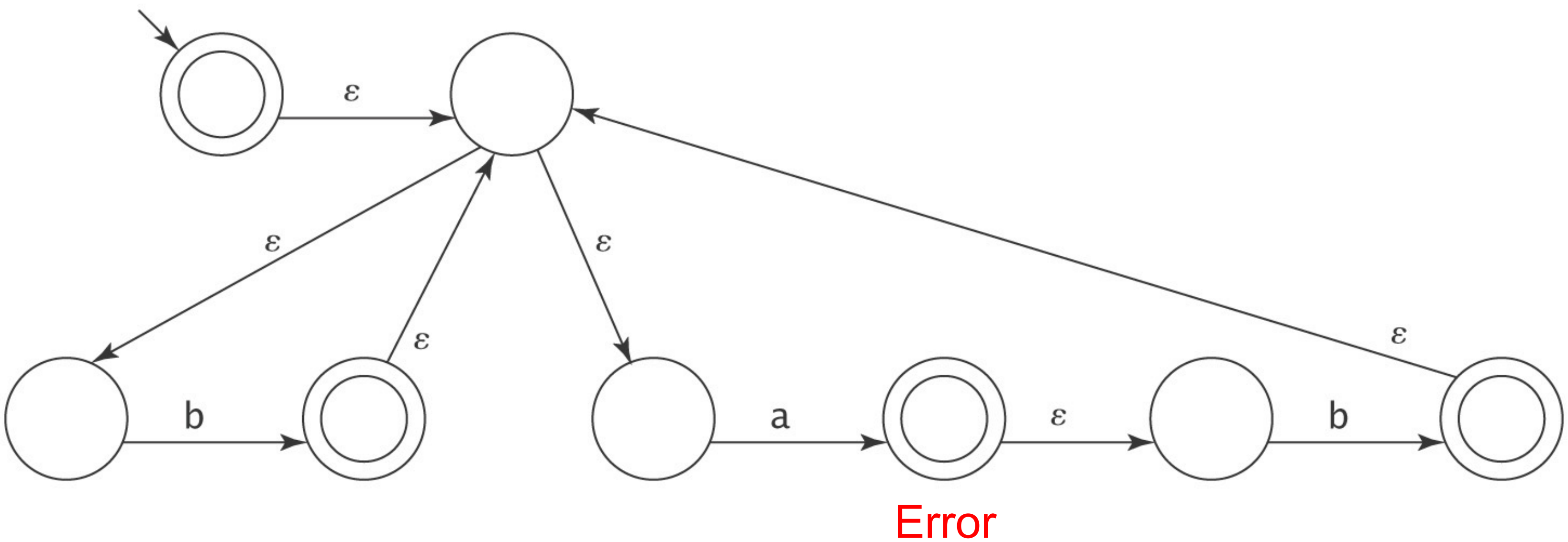
An FSM for (b ∪ ab):



An Example

(b ∪ ab)*

An FSM for (b ∪ ab)*:



The Algorithm *regextofsm*

regextofsm(α: regular expression) =

Beginning with the primitive subexpressions of α and working outwards until an FSM for all of α has been built do:

Construct an FSM as described above.

For Every FSM There is a Corresponding Regular Expression

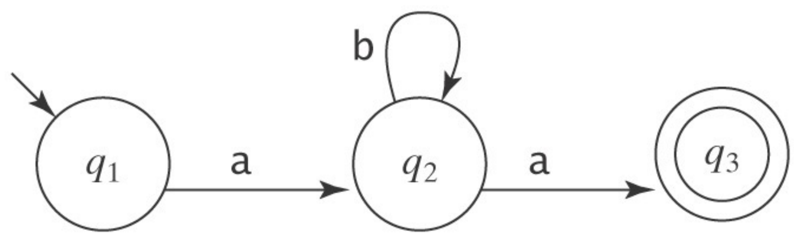
We'll show this by construction.

The key idea is that we'll allow arbitrary regular expressions to label the transitions of an FSM.

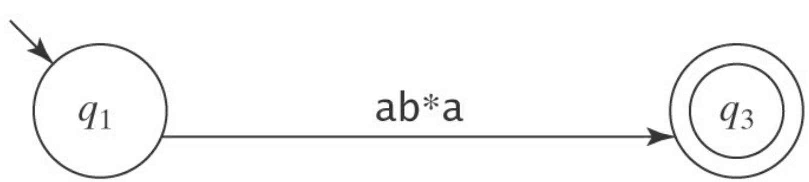
Read if interested …

A Simple Example

Let *M* be:



Suppose we rip out state 2:



The Algorithm *fsmtoregexheuristic*

fsmtoregexheuristic(*M*: FSM) =

1. Remove unreachable states from *M*.
2. If *M* has no accepting states then return ∅.
3. If the start state of *M* is part of a loop, create a new start state *s* and connect *s* to *M*' s start state via an ε-transition.
4. If there is more than one accepting state of *M* or there are any transitions out of any of them, create a new accepting state and connect each of *M*' s accepting states to it via an ε-transition. The old accepting states no longer accept.
5. If *M* has only one state then return ε.
6. Until only the start state and the accepting state remain do:
    - 6.1 Select *rip* (not *s* or an accepting state).
    - 6.2 Remove *rip* from *M*.
    - 6.3 *Modify the transitions among the remaining states so *M* accepts the same strings.
7. Return the regular expression that labels the one remaining transition from the start state to the accepting state.

Regular Expressions in Perl

Syntax	Name	Description
abc	Concatenation	Matches a, then b, then c, where a, b, and c are any regexs
a | b | c	Union (Or)	Matches a or b or c, where a, b, and c are any regexs
a*	Kleene star	Matches 0 or more a' s, where a is any regex
a+	At least one	Matches 1 or more a' s, where a is any regex
a?		Matches 0 or 1 a' s, where a is any regex
a{n, m}	Replication	Matches at least n but no more than m a' s, where a is any regex
a*?	Parsimonious	Turns off greedy matching so the shortest match is selected
a+?	″	″
.	Wild card	Matches any character except newline
^	Left anchor	Anchors the match to the beginning of a line or string
$	Right anchor	Anchors the match to the end of a line or string
[a-z]		Assuming a collating sequence, matches any single character in range
[^a-z]		Assuming a collating sequence, matches any single character not in range
\d	Digit	Matches any single digit, i.e., string in [0-9]
\D	Nondigit	Matches any single nondigit character, i.e., [^0-9]
\w	Alphanumeric	Matches any single "word" character, i.e., [a-zA-Z0-9]
\W	Nonalphanumeric	Matches any character in [^a-zA-Z0-9]
\s	White space	Matches any character in [space, tab, newline, etc.]

Regular Expressions in Perl

Syntax	*Name*	*Description*
\S	Nonwhite space	Matches any character not matched by \s
\n	Newline	Matches newline
\r	Return	Matches return
\t	Tab	Matches tab
\f	Formfeed	Matches formfeed
\b	Backspace	Matches backspace inside []
\b	Word boundary	Matches a word boundary outside []
\B	Nonword boundary	Matches a non-word boundary
\0	Null	Matches a null character
nnn	Octal	Matches an ASCII character with octal value *nnn*
\x*nn*	Hexadecimal	Matches an ASCII character with hexadecimal value *nn*
\c*X*	Control	Matches an ASCII control character
char	Quote	Matches *char*; used to quote symbols such as . and \
(*a*)	Store	Matches *a*, where *a* is any regex, and stores the matched string in the next variable
\1	Variable	Matches whatever the first parenthesized expression matched
\2		Matches whatever the second parenthesized expression matched
…		For all remaining variables

Testing. many other online tools

Using Regular Expressions in the Real World

Matching numbers:

-?([0-9]+(\.[0-9]*)?|\.[0-9]+)

Matching ip addresses:

[0-9]{1,3}(\.[0-9]{1,3}){3}

Trawl for email addresses:

```
\b[A-Za-z0-9_%-]+@[A-Za-z0-9_%-]+ (\.[A-Za-
  z]+){1,4}\b
```

From Friedl, J., Mastering Regular Expressions, O' Reilly,1997.

IE: information extraction, unstructured data management

A Biology Example – BLAST

Given a protein or DNA sequence, find others that are likely to be evolutionarily close to it.

ESGHDTTTYYNKNRYPAGWNNHHDQMFFWV

Build a DFSM that can examine thousands of other sequences and find those that match any of the selected patterns.

Simplifying Regular Expressions

Regex's describe sets:

- Union is commutative: $\alpha \cup \beta = \beta \cup \alpha$.
- Union is associative: $(\alpha \cup \beta) \cup \gamma = \alpha \cup (\beta \cup \gamma)$.
- $\varnothing$ is the identity for union: $\alpha \cup \varnothing = \varnothing \cup \alpha = \alpha$.
- Union is idempotent: $\alpha \cup \alpha = \alpha$.

Concatenation:

- Concatenation is associative: $(\alpha\beta)\gamma = \alpha(\beta\gamma)$.
- ε is the identity for concatenation: $\alpha \varepsilon = \varepsilon \alpha = \alpha$.
- $\varnothing$ is a zero for concatenation: $\alpha \varnothing = \varnothing \alpha = \varnothing$.

Concatenation distributes over union:

- $(\alpha \cup \beta) \gamma = (\alpha \gamma) \cup (\beta \gamma)$.
- $\gamma (\alpha \cup \beta) = (\gamma \alpha) \cup (\gamma \beta)$.

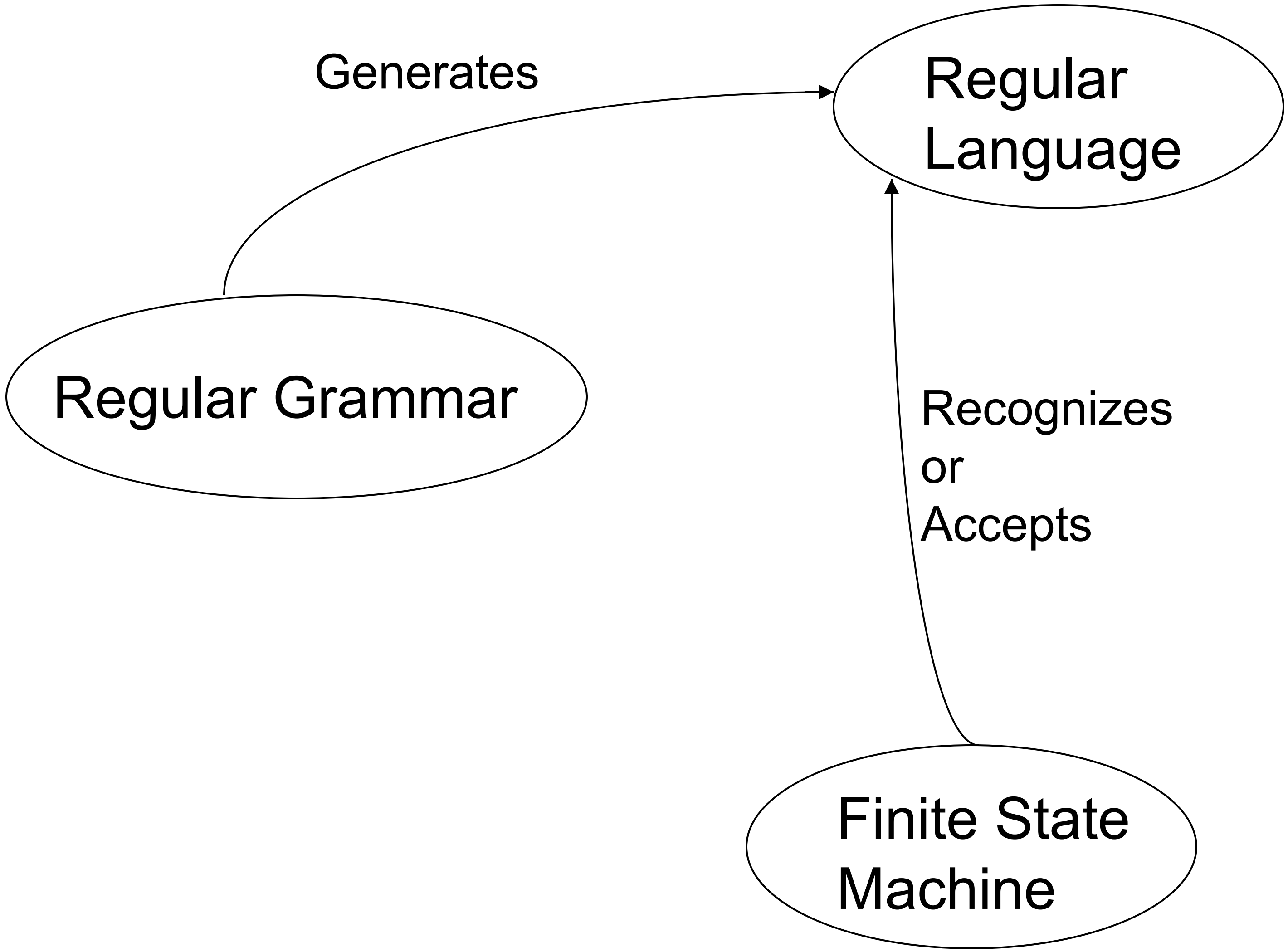
Kleene star:

- $\varnothing^* = \varepsilon$.
- $\varepsilon^* = \varepsilon$.
- $(\alpha^*)^* = \alpha^*$.
- $\alpha^*\alpha^* = \alpha^*$.
- $\alpha \cup \beta)^* = (\alpha^*\beta^*)^*$.

Regular Grammars

Chapter 7

Regular Languages



Regular Grammars

A regular grammar *G* is a quadruple (*V*, Σ, *R*, S), where:

- *V* (rule alphabet) contains nonterminals and terminals
  - terminals: symbols that can appear in strings generated by *G*
  - nonterminals: symbols that are used in the grammar but do not appear in strings of the language
- Σ (the set of terminals) is a subset of *V*,
- *R* (the set of rules) is a finite set of rules of the form:

$$X \rightarrow Y$$

- *S* (the start symbol) is a nonterminal

Regular Grammars

In a regular grammar, all rules in R must:

- have a left hand side that is a single nonterminal
- have a right hand side that is:
  ε, or a single terminal, or a single terminal followed by a single nonterminal.

Legal: $S \rightarrow \texttt{a}$, $S \rightarrow \varepsilon$, and $T \rightarrow \texttt{a}S$

Not legal: $S \rightarrow \texttt{a}S\texttt{a}$ and $\texttt{a}S\texttt{a} \rightarrow T$

- Regular grammars must always produce strings one character at a time, moving left to right.

Regular Grammars

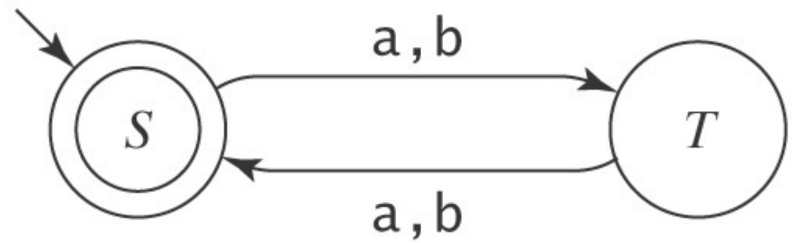
- The one we study is actually ***right regular grammar***.
  - Also called right linear grammar
  - Generates regular languages, recognized by FSM
  - Note FSM reads the input string *w* left to right

- ***Left regular grammar*** (left linear grammar)
  - $S \rightarrow \texttt{a}$, $S \rightarrow \varepsilon$, and $T \rightarrow S\texttt{a}$
  - Does it generate regular languages?

Regular Grammar Example

$L = \{w \in \{a, b\}^* : |w| \text{ is even}\}$ $((aa) \cup (ab) \cup (ba) \cup (bb))^*$

M:



G:

$S \rightarrow \varepsilon$

$S \rightarrow aT$

$S \rightarrow bT$

$T \rightarrow aS$

$T \rightarrow bS$

- By convention, the start symbol of any grammar G will be the symbol on the left-hand side of the first rule
- **Notice the clear correspondence between *M* and *G***
  - Given one, easy to derive the other
  - Works for DFSM, and NDFSM without ε-transitions
  - T or F: For any NDFSM M, we can find M', an NDFSM without ε-transitions, such that L(M) = L(M').

Regular Languages and Regular Grammars

Theorem: The class of languages that can be defined with regular grammars is exactly the regular languages.

Proof: By two constructions.

Regular Languages and Regular Grammars

Regular grammar* $\rightarrow$ *FSM:

grammartofsm($G = (V, \Sigma, R, S)$) =

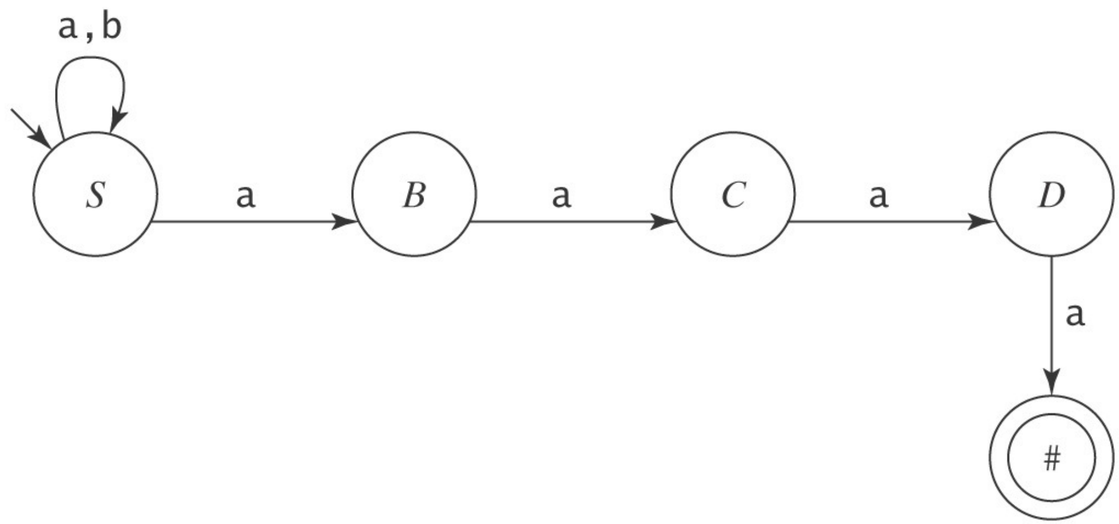
1. Create in M a separate state for each nonterminal in V.
2. Start state is the state corresponding to S .
3. If there are any rules in R of the form $X \rightarrow w$, for some $w \in \Sigma$, create a new state labeled #.
4. For each rule of the form $X \rightarrow w\ Y$, add a transition from X to Y labeled w.
5. For each rule of the form $X \rightarrow w$, add a transition from X to # labeled w.
6. For each rule of the form $X \rightarrow \varepsilon$, mark state X as accepting.
7. Mark state # as accepting.

FSM* $\rightarrow$ *Regular grammar: Similarly.

Strings That End with aaaa

$L = \{w \in \{\texttt{a}, \texttt{b}\}^* : w$ ends with the pattern $\texttt{aaaa}\}$.

$S \rightarrow \texttt{a}S$
$S \rightarrow \texttt{b}S$
$S \rightarrow \texttt{a}B$
$B \rightarrow \texttt{a}C$
$C \rightarrow \texttt{a}D$
$D \rightarrow \texttt{a}$



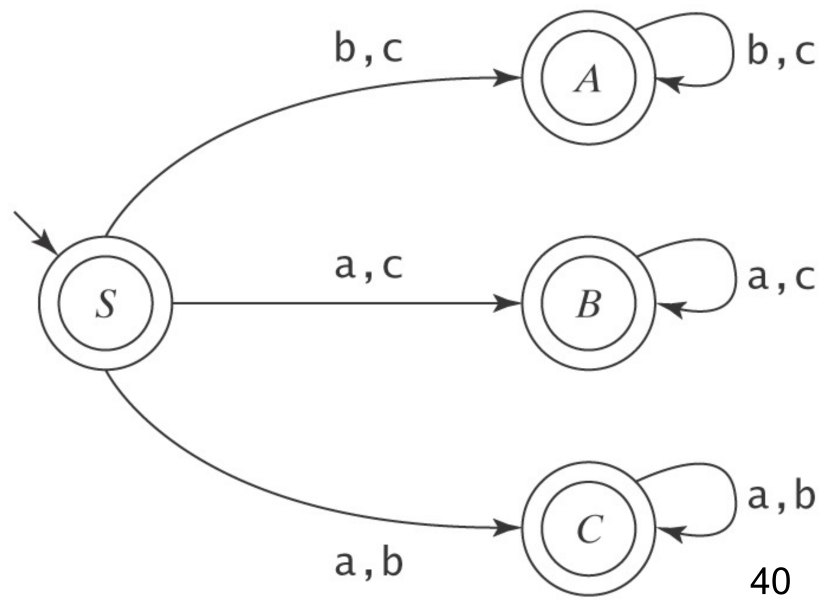
One Character Missing

$L = \{w \in \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}^*$: there is a symbol in the alphabet not appearing in $w\}$.

$S \rightarrow \varepsilon$
$S \rightarrow \mathtt{a}B$
$S \rightarrow \mathtt{a}C$
$S \rightarrow \mathtt{b}A$
$S \rightarrow \mathtt{b}C$
$S \rightarrow \mathtt{c}A$
$S \rightarrow \mathtt{c}B$

$A \rightarrow \mathtt{b}A$
$A \rightarrow \mathtt{c}A$
$A \rightarrow \varepsilon$
$B \rightarrow \mathtt{a}B$
$B \rightarrow \mathtt{c}B$
$B \rightarrow \varepsilon$

$C \rightarrow \mathtt{a}C$
$C \rightarrow \mathtt{b}C$
$C \rightarrow \varepsilon$



ε Transitions are removed.